2-ADIC BEHAVIOR OF NUMBERS OF DOMINO TILINGS

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Dedicated to my grandparents Garnette Cohn (1907–1998) and Lee Cohn (1908–1998)

Abstract. We study the 2-adic behavior of the number of domino tilings of a $2n \times 2n$ square as n varies. It was previously known that this number was of the form $2^n f(n)^2$, where $f(n)$ is an odd, positive integer. We show that the function f is uniformly continuous under the 2-adic metric, and thus extends to a function on all of $\mathbb{Z}$. The extension satisfies the functional equation $f(-1-n) = \pm f(n)$, where the sign is positive iff $n \equiv 0, 3 \pmod 4$.

Kasteleyn [K], and Temperley and Fisher [TF], proved that the number of tilings of a $2n \times 2n$ square with 1×2 dominos is

$$\prod_{i=1}^{n}\prod_{j=1}^{n}\left(4\cos^2\frac{\pi i}{2n+1} + 4\cos^2\frac{\pi j}{2n+1}\right).$$

Although it is by no means obvious at first glance, this number is always a perfect square or twice a perfect square (see [L]). Furthermore, it is divisible by 2^n but no higher power of 2. This fact about 2-divisibility was independently proved by several people (see [JSZ], or see [P] for a combinatorial proof), but there seems to have been little further investigation of the 2-adic properties of these numbers, except for [JS].

Write the number of tilings as $2^n f(n)^2$, where $f(n)$ is an odd, positive integer. In this paper, we study the 2-adic properties of the function f. In particular, we will prove the following theorem, which was conjectured by James Propp:

Theorem 1. *The function f is uniformly continuous under the 2-adic metric, and its unique extension to a function from $\mathbb{Z}_2$ to $\mathbb{Z}_2$ satisfies the functional equation*

$$f(-1-n) = \begin{cases} f(n) & \text{if } n \equiv 0, 3 \pmod 4\text{, and} \\ -f(n) & \text{if } n \equiv 1, 2 \pmod 4. \end{cases}$$

John and Sachs [JS] have independently investigated the 2-adic behavior of f, and explicitly determined it modulo 2^6. Their methods, as well as ours, can be used to write formulas for f modulo any power of 2, but no closed form is known.

The proof of Theorem 1 will not make any use of sophisticated 2-adic machinery. The only non-trivial fact we will require is that the 2-adic absolute value extends uniquely to each finite extension of $\mathbb{Q}$. For this fact, as well as basic definitions and concepts, the book [G] by Gouvêa is an excellent reference.

It is helpful to keep in mind this more elementary description of what it means for f to be uniformly continuous 2-adically: for every k, there exists an ℓ such that if $n \equiv m \pmod{2^\ell}$, then $f(n) \equiv f(m) \pmod{2^k}$. In particular, we will see that for our function f, the condition $n \equiv m \pmod 2$ implies that $f(n) \equiv f(m) \pmod 2$, and $n \equiv m \pmod 4$ implies that $f(n) \equiv f(m) \pmod 4$.

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As a warm-up in using 2-adic methods, and for the sake of completeness, we will prove that that number of tilings of a $2n \times 2n$ square really is of the form $2^n f(n)^2$, assuming Kasteleyn's theorem. To do so, we will make use of the fact that the 2-adic metric extends to every finite extension of $\mathbb{Q}$, in particular the cyclotomic extensions, which contain the cosines that appear in Kasteleyn's product formula. We can straightforwardly determine the 2-adic valuation of each factor, and thus of the entire product.

Let ζ be a primitive $(2n+1)$-st root of unity, and define

$$\alpha_{i,j} = \zeta^i + \zeta^{-i} + \zeta^j + \zeta^{-j}.$$

Then the number of domino tilings of a $2n \times 2n$ square is

$$\prod_{i=1}^{n} \prod_{j=1}^{n} (4 + \alpha_{i,j}). \tag{1}$$

To determine the divisibility by 2, we look at this number as an element of $\mathbb{Q}_2(\zeta)$. Because $2n+1$ is odd, the extension $\mathbb{Q}_2(\zeta)/\mathbb{Q}_2$ is unramified, so 2 remains prime in $\mathbb{Q}_2(\zeta)$. We will use $|\cdot|_2$ to denote the unique extension of the 2-adic absolute value to $\mathbb{Q}_2(\zeta)$.

Lemma 2. *For $1 \le i, j \le n$, we have*

$$|4 + \alpha_{i,j}|_2 = \begin{cases} 1 & \text{if } i \ne j, \text{ and} \\ 1/2 & \text{if } i = j. \end{cases}$$

Proof. The number $4 + \alpha_{i,j}$ is an algebraic integer, so its 2-adic absolute value is at most 1. To determine how much smaller it is, first notice that

$$\alpha_{i,j} = (\zeta^i + \zeta^j)(\zeta^{i+j} + 1)\zeta^{-i}\zeta^{-j}.$$

In order for $4 + \alpha_{i,j}$ to reduce to 0 modulo 2, we must have

$$\zeta^i \equiv \zeta^{\pm j} \pmod{2}.$$

However, this is impossible unless $i \equiv \pm j \pmod{2n+1}$, because ζ has order $2n+1$ in the residue field. Since $1 \le i, j \le n$, the only possibility is $i = j$.

In that case, $4 + \alpha_{i,i} = 2(2 + \zeta^i + \zeta^{-i})$. In order to have $|4 + \alpha_{i,i}|_2 < 1/2$, the second factor would need to reduce to 0. However, that could happen only if $\zeta^i \equiv \zeta^{-i} \pmod{2}$, which is impossible. $\square$

By Lemma 2, the product (1) is divisible by 2^n but not 2^{n+1}. The product of the terms with $i = j$, divided by 2^n, is

$$\prod_{i=1}^{n} (2 + \zeta^i + \zeta^{-i}), \tag{2}$$

which equals 1, as we can prove by writing

$$\prod_{i=1}^{n} (2 + \zeta^i + \zeta^{-i}) = \prod_{i=1}^{n} (1 + \zeta^i)(1 + \zeta^{-i}) = \prod_{i=1}^{n} (1 + \zeta^i)(1 + \zeta^{2n+1-i}) = \prod_{i=1}^{2n} (1 + \zeta^i) = 1;$$

the last equality follows from substituting $z = -1$ in

$$z^{2n+1} - 1 = \prod_{i=0}^{2n} (z - \zeta^i).$$

Thus, the odd factor of the number of tilings of a $2n \times 2n$ square is

$$f(n)^2 = \prod_{1 \le i < j \le n} (4 + \alpha_{i,j})^2.$$

We are interested in the square root of this quantity, not the whole odd factor. The positive square root is

$$f(n) = \prod_{1 \le i < j \le n} (4 + \alpha_{i,j})$$

(notice that every factor is positive). It is clearly an integer, since it is an algebraic integer and is invariant under every automorphism of $\mathbb{Q}(\zeta)/\mathbb{Q}$. Thus, we have shown that the number of tilings is of the form $2^n f(n)^2$, where $f(n)$ an odd integer.

In determining the 2-adic behavior of f, it seems simplest to start by examining it modulo 4. In that case, we have the formula

$$f(n) \equiv \prod_{1 \le i < j \le n} \alpha_{i,j} \pmod 4,$$

and the product appearing in it can actually be evaluated explicitly.

Lemma 3. *We have*

$$\prod_{1 \le i < j \le n} \alpha_{i,j} = \begin{cases} 1 & \text{if } n \equiv 0, 1, 3 \pmod 4, \text{ and} \\ -1 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. In this proof, we will write ζ^* to indicate an unspecified power of ζ. Because the product in question is real and the only real power of ζ is 1, we will in several cases be able to see that factors of ζ^* equal 1 without having to count the ζ's.

Start by observing that

$$\begin{aligned} \prod_{1 \le i < j \le n} \alpha_{i,j} &= \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (\zeta^{i+j} + 1)(\zeta^{i-j} + 1)\zeta^{-i} \\ &= \zeta^* \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} (\zeta^{i+j} + 1)(\zeta^{2n+1+i-j} + 1) \\ &= \zeta^* \prod_{i=1}^{n-1} \prod_{s=2i+1}^{2n} (\zeta^s + 1). \end{aligned}$$

(To prove the last line, check that $i + j$ and $2n + 1 + i - j$ together run over the same range as s.)

In the factors where $i > n/2$, replace $\zeta^s + 1$ with $\zeta^s(\zeta^{2n+1-s} + 1)$. Now for every i, it is easy to check that

$$\prod_{s=2i+1}^{2n} (\zeta^s + 1) \prod_{s=2(n-i)+1}^{2n} (\zeta^{2n+1-s} + 1) = \prod_{s=1}^{2n} (\zeta^s + 1) = 1.$$

When n is odd, pairing i with $n - i$ in this way takes care of every factor except for a power of ζ, which must be real and hence 1. Thus, the whole product is 1 when n is odd, as desired.

In the case when n is even, the pairing between i and $n - i$ leaves the $i = n/2$ factor unpaired. The product is thus

$$\zeta^* \prod_{s=n+1}^{2n} (\zeta^s + 1). \tag{3}$$

Notice that

$$\begin{aligned}\left(\prod_{s=n+1}^{2n}(1+\zeta^s)\right)^2 &= \prod_{s=n+1}^{2n}\zeta^s(1+\zeta^{2n+1-s})\prod_{s=n+1}^{2n}(1+\zeta^s)\\ &= \prod_{s=n+1}^{2n}\zeta^s\\ &= \zeta^*.\end{aligned}$$

Hence, since every power of ζ has a square root among the powers of ζ (because $2n+1$ is odd),

$$\prod_{s=n+1}^{2n}(\zeta^s+1) = \pm\zeta^*.$$

Substituting this result into (3) shows that the product we are trying to evaluate must equal ± 1, since the ζ^* factor must be real and therefore 1. All that remains is to determine the sign.

Since

$$\prod_{s=n+1}^{2n}(1+\zeta^s)$$

and

$$\prod_{t=1}^{n}(1+\zeta^t)$$

are reciprocals, it is enough to answer the question for the second one (which is notationally slightly simpler). We know that it is plus or minus a power of ζ, and need to determine which. Since $\zeta = \zeta^{-2n}$, we have

$$\prod_{t=1}^{n}(1+\zeta^t) = \prod_{t=1}^{n}(1+\zeta^{-2nt}) = \zeta^*\prod_{t=1}^{n}(\zeta^{nt}+\zeta^{-nt}).$$

The product

$$\prod_{t=1}^{n}(\zeta^{nt}+\zeta^{-nt})$$

is real, so it must be ± 1; to determine which, we just need to determine its sign. For that, we write

$$\zeta^{nt}+\zeta^{-nt} = 2\cos\left(t\pi - \frac{t\pi}{2n+1}\right),$$

which is negative iff t is odd (assuming $1 \le t \le n$). Thus, the sign of the product is negative iff there are an odd number of odd numbers from 1 to n, i.e., iff $n \equiv 2 \pmod 4$ (since n is even).

Therefore, the whole product is -1 iff $n \equiv 2 \pmod 4$, and is 1 otherwise. □

Now that we have dealt with the behavior of f modulo 4, we can simplify the problem considerably by working with f^2 rather than f. Recall that proving uniform continuity is equivalent to showing that for every k, there exists an ℓ such that if $n \equiv m \pmod{2^\ell}$, then $f(n) \equiv f(m) \pmod{2^k}$. If we can find an ℓ such that $n \equiv m \pmod{2^\ell}$ implies that $f(n)^2 \equiv f(m)^2 \pmod{2^{2k}}$, then it follows that $f(n) \equiv \pm f(m) \pmod{2^k}$, and our knowledge of f modulo 4 pins down the sign as $+1$. The same reasoning applies to the functional equation, so if we can show that f^2 is uniformly continuous 2-adically and satisfies $f(-1-n)^2 = f(n)^2$, then we will have proved Theorem 1.

We begin by using (1) to write

$$\begin{aligned} 2^n f(n)^2 &= \left(\prod_{i,j=1}^{n} \alpha_{i,j}\right) \prod_{i,j=1}^{n} \left(1 + \frac{4}{\alpha_{i,j}}\right) \\ &= \left(\prod_{i,j=1}^{n} \alpha_{i,j}\right) \sum_{k \geq 0} 4^k E_k(n), \end{aligned}$$

where $E_k(n)$ is the k-th elementary symmetric polynomial in the $1/\alpha_{i,j}$'s (where $1 \leq i, j \leq n$). We can evaluate the product

$$\prod_{i,j=1}^{n} \alpha_{i,j}$$

by combining Lemma 3 with the equation

$$\prod_{t=1}^{n} (\zeta^t + \zeta^{-t}) = (-1)^{\lfloor \frac{n+1}{2} \rfloor},$$

which can be proved using the techniques of Lemma 3: it is easily checked that the product squares to 1, and its sign is established by writing

$$\zeta^t + \zeta^{-t} = 2 \cos \frac{2t\pi}{2n+1},$$

which is positive for $1 \leq t < (2n+1)/4$ and negative for $(2n+1)/4 < t \leq n$. This shows that

$$\prod_{i,j=1}^{n} \alpha_{i,j} = (-1)^{\lfloor \frac{n+1}{2} \rfloor} 2^n,$$

so we conclude that

$$f(n)^2 = (-1)^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k \geq 0} 4^k E_k(n). \tag{4}$$

The function $n \mapsto (-1)^{\lfloor \frac{n+1}{2} \rfloor}$ is uniformly continuous 2-adically and invariant under interchanging n with $-1-n$, so to prove these properties for f^2 we need only prove them for the sum on the right of (4).

Because $\alpha_{i,j}$ has 2-adic valuation at most 1, that of $E_k(n)$ is at least $-k$, and hence $2^k E_k(n)$ is a 2-adic integer (in the field $\mathbb{Q}_2(\zeta)$). Thus, to determine $f(n)^2$ modulo 2^k we need only look at the first $k+1$ terms of the sum (4).

Define

$$S_k(n) = \sum_{i,j=1}^{n} \frac{1}{\alpha_{i,j}^k}.$$

We will prove the following proposition about S_k.

Proposition 4. *For each k, $S_k(n)$ is a polynomial over $\mathbb{Q}$ in n and $(-1)^n$. Furthermore,*

$$S_k(n) = S_k(-1-n).$$

We will call a polynomial in n and $(-1)^n$ a *quasi-polynomial*. Notice that every quasi-polynomial over $\mathbb{Q}$ is uniformly continuous 2-adically.

In fact, S_k is actually a polynomial of degree $2k$. However, we will not need to know that. The only use we will make of the fact that S_k is a quasi-polynomial is in proving uniform continuity, so we will prove only this weaker claim.

Given Proposition 4, the same must hold for E_k, because the E_k's and S_k's are related by the Newton identities

$$kE_k = \sum_{i=1}^{k} (-1)^{i-1} S_i E_{k-i}.$$

It now follows from (4) that f^2 is indeed uniformly continuous and satisfies the functional equation. Thus, we have reduced Theorem 1 to Proposition 4.

Define

$$T_k(n) = \sum_{i,j=0}^{2n} \frac{1}{\alpha_{i,j}^k},$$

and

$$R_k(n) = \sum_{i=0}^{2n} \frac{1}{\alpha_{i,0}^k}.$$

Because $\alpha_{i,j} = \alpha_{-i,j} = \alpha_{i,-j} = \alpha_{-i,-j}$, we have

$$T_k(n) = 4S_k(n) + 2R_k(n) - \frac{1}{\alpha_{0,0}^k}.$$

To prove Proposition 4, it suffices to prove that T_k and R_k are quasi-polynomials over $\mathbb{Q}$, and that $T_k(-1-n) = T_k(n)$ and $R_k(-1-n) = R_k(n)$.

We can simplify further by reducing T_k to a single sum, as follows. It is convenient to write everything in terms of roots of unity, so that

$$T_k(n) = \sum_{\zeta,\xi} \frac{1}{(\zeta + 1/\zeta + \xi + 1/\xi)^k},$$

where ζ and ξ range over all $(2n+1)$-st roots of unity. (This notation supersedes our old use of ζ.) Then we claim that

$$T_k(n) = \left(\sum_{\zeta} \frac{1}{(\zeta + 1/\zeta)^k} \right)^2.$$

To see this, write the right hand side as

$$\left(\sum_{\zeta} \frac{1}{(\zeta + 1/\zeta)^k} \right) \left(\sum_{\xi} \frac{1}{(\xi + 1/\xi)^k} \right) = \sum_{\zeta,\xi} \frac{1}{(\zeta\xi + 1/(\zeta\xi) + \zeta/\xi + 1/(\zeta/\xi))^k},$$

and notice that as ζ and ξ run over all $(2n+1)$-st roots of unity, so do $\zeta\xi$ and ζ/ξ. (This is equivalent to the fact that every $(2n+1)$-st root of unity has a unique square root among such roots of unity, because that implies that the ratio ξ^2 between $\zeta\xi$ and ζ/ξ does in fact run over all $(2n+1)$-st roots of unity.)

We can deal with R_k similarly: as ξ runs over all $(2n+1)$-st roots of unity, so does ξ^2, and hence

$$R_k(n) = \sum_{\zeta} \frac{1}{(2 + \zeta + 1/\zeta)^k} = \sum_{\xi} \frac{1}{(2 + \xi^2 + 1/\xi^2)^k} = \sum_{\xi} \frac{1}{(\xi + 1/\xi)^{2k}}.$$

Define

$$U_k(n) = \sum_{\zeta} \frac{1}{(\zeta + 1/\zeta)^k}.$$

Now everything comes down to proving the following proposition:

Proposition 5. *The function U_k is a quasi-polynomial over $\mathbb{Q}$, and satisfies*

$$U_k(-1-n) = U_k(n).$$

Proof. The proof is based on the observation that for any non-zero numbers, the power sums of their reciprocals are minus the Taylor coefficients of the logarithmic derivative of the polynomial with those numbers as roots, i.e.,

$$\begin{aligned}\frac{d}{dx}\log\prod_{i=1}^{m}(x-r_i) &= \sum_{i=1}^{m}\frac{1}{x-r_i}\\ &= \sum_{i=1}^{m}\frac{-1/r_i}{1-x/r_i}\\ &= -\sum_{i=1}^{m}\left(\frac{1}{r_i}+\frac{x}{r_i^2}+\frac{x^2}{r_i^3}+\dots\right).\end{aligned}$$

To apply this fact to U_k, define

$$\begin{aligned}P_n(x) &= \prod_{\zeta}(x-(\zeta+1/\zeta))\\ &= \prod_{j=0}^{2n}(x-2\cos(2\pi j/(2n+1)))\\ &= 2(\cos((2n+1)\cos^{-1}(x/2))-1).\end{aligned}$$

Then

$$\frac{d}{dx}\log P_n(x) = \frac{2n+1}{2\sqrt{1-x^2/4}}\frac{\sin((2n+1)\cos^{-1}(x/2))}{\cos((2n+1)\cos^{-1}(x/2))-1}.$$

This function is invariant under interchanging n with $-1-n$ (equivalently, interchanging $2n+1$ with $-(2n+1)$), so its Taylor coefficients are as well. By the observation above, the coefficient of x^k is $-U_{k+1}(n)$. Straightforward calculus shows that these coefficients are polynomials over $\mathbb{Q}$ in n, $\sin((2n+1)\pi/2)$, and $\cos((2n+1)\pi/2)$. Using the fact that $\cos((2n+1)\pi/2)=0$ and $\sin((2n+1)\pi/2)=(-1)^n$ completes the proof. ☐

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